Measuring Fractions

Annie Mitchell
Bethal Primary School
<mitchell annie e@edumail vic gov au >

Twenty-six children from Years 5 to 8 were given a task-based interview on fraction knowledge The verbal responses of the children to three tasks out of the 35 offered are discussed in depth here The use of measurement and spatial language was evident in correct and incorrect answers What was more critical were misconceptions about understandings of measurement principles The rational number domain with its infinite number of points between any two fractions can be seen as more like the measurement domain where continuous properties are measured, than the whole number domain where discrete items are counted This analogy may be useful for children's conceptual understanding of fractions

When children encounter fractions and decimals, they initially begin to operate in a number system which is continuous, as there is an infinite number of rational numbers between two points on a number line Counting discrete objects, as a strategy, will not be enough in this new universe And yet the part-whole aspect of fractional understanding is often taught as counting and matching Children are introduced to fractions by being shown a set of elements, for example a rectangle divided into three equal pieces, some of which are shaded, and told to count The cardinal number of the count of shaded objects is the numerator, while the cardinal number of the sum shaded and unshaded elements becomes the denominator And thus fractions become tied to counting and matching (Carrahar, 1996) Importantly, this strategy is not helpful when the whole is divided into unequal-sized pieces

The action of counting is to count discrete objects but to measure is to measure continuous properties such as length, area or volume (Wilson & Rowland, 1993) In order to work with a part-whole area representation with unequal parts, the parts need to be measured not counted This brings up two different but related issues Firstly, teachers often use area models or length models in fraction tasks without ascertaining that each child is confident in the applications of measurement that they will need to use as a tool to help them think conceptually about the task at hand Secondly, and more importantly, seeing fraction manipulations as guided by measurement principles may be more powerful than just using whole number algorithmic procedures Fraction density, the concept that between any two fractions there are an infinite number of rational numbers, may be more easily understood when thought of in measurement terms rather than in whole number terms

In a framework for the teaching of early measurement, the principles of measurement are articulated; students need to identify the attribute, need to know that the quantity is unchanged if it is rearranged (conservation), that lengths, areas, and volumes can be subdivided into equal parts (units), and iteration of units involves no overlapping and no gaps (Outhred & McPhail, 2000) Studies of conservation of length, area, and volume have concentrated traditionally on identifying the age and stage that young children attain conservation of an attribute of measure (Carpenter, 1975; Carpenter & Lewis, 1976) While some researchers see conservation of length and area as posing some difficulty for middle primary children, others, on the other hand, see congruence as a concept mastered early in the sequence of partitioning and sharing (Pothier & Sawada, 1983)

Kieren detailed sub-constructs of rational number: decimals, equivalent fractions, ratio numbers, multiplicative operators, quotients, and measures (Kieren, 1976) A foundational idea in the teaching of measurement is the concept of the unit "There is a unit that can be assigned the number 1 The unit must be compatible with the property being measured " (Wilson & Osbourne, 1992) When measuring objects, a unit is chosen first and iterated to measure a property and there may be part of a unit left over at the end When dividing a whole into parts, the parts (e g quarters) are adjusted until the set number of equal parts leave no remainder (without gaps and without overlapping) Kieren's concept of measures as a sub-construct of rational number and choosing a unit to measure are important distinctions

The researchers in the Rational Number Project used Kieren's framework (Behr & Post, 1992) In order to assess and teach for conceptual understanding, the Rational Number Project team created activities and task-based assessment interviews that probed understanding rather than procedural proficiency Their teaching tasks include paper folding Their assessment tasks include area models with perceptual distracters, where pre-drawn partitioning is counter-intuitive, for example a child may be asked to show a third of a rectangle which has been pre-divided in half (Cramer, Post & Behr, 1989; Cramer, Post & delMas, 2002) Similar part-whole tasks, e g a square is segmented into quarters and then one of those quarters is further segmented to give five (or six or seven) pieces (see e g Saxe, Taylor, McIntosh & Gearhart, 2005) have also been used by other researchers

The Present Study

The purpose of the study was to pilot a collection of informative tasks to be used in a one-to-one, task-based interview (see e g , Mitchell & Clarke, 2004) assessing children's conceptual understanding of fractions and decimals Although the sample size was small, (*n* =26), interesting understandings and misconceptions emerged from the children's articulation of how they worked at each task

The participants were 18 Year 5 children from a low socio economic, co-educational state primary school in suburban Melbourne and 8 students from a middle class, co-educational state secondary college also in suburban Melbourne The students in Year 5 came from several different classes and represented a wide range of achievement in mathematics The five Year 8 and three Year 10 students were chosen by their teachers as having above average achievement in mathematics The Year 5 children were interviewed at the end of the school year, while the Year 8 and 10 students were interviewed at the start of the school year

The tasks in the fraction interview were divided into seven sections; part-whole, connecting concepts with symbols/equivalence, fractions as a number, fractions as division, relative size/benchmarking, operations and operators, and proportional reasoning The Year 5 children were generally given all 35 tasks in the fraction interview If a child, however, was having obvious difficulty, not all questions were offered, but as the children were not told whether they were correct or incorrect following their initial answer and were always asked to explain their reasoning, they were not as acutely aware of "doing well" or "doing badly" The eight students in the secondary setting were offered as much of the fraction interview as possible in an approximately 43 minute period The Year 5 children were also given a 14 task decimal interview (Roche & Clarke, 2004; Roche, in press)

In this paper, three of the 49 fraction and decimal tasks will be considered in depth Task 1 (Question 2 of the fraction interview; see Figure 1) was chosen partly for its affective value, as it gave the children a chance to fold paper into quarters as a nonthreatening introduction to the task based interview In each of the written versions of the three tasks, italics indicates what the interviewer does (or what the child may say or do), and normal text indicates the text of the interview, or the exact script that the researcher follows

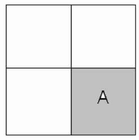
Task 1, Fold Me a Quarter

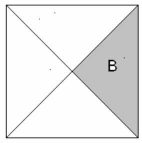
Hand the student one Kinder/Brenex square

a) Please fold the square into quarters

Hand the student a second Kinder/Brenex square

b) Please fold this into quarters another way





Show the student two squares already folded into quarters (in squares and triangles- with a quarter of each shaded, with a square marked A and a triangle marked B)

c) Here are two squares that I have already folded Imagine that these are sandwiches, cut into four parts like these Which shaded part, A (*square*) or B (*triangle)* would give you more? …… Explain how you know

Figure 1 Task 1, Fold Me a Quarter

The record sheet included spaces to indicate the way in which students folded the paper; squares, triangles (as above), rectangles, or "other" (many possibilities but not so easy to fold)

Task 2 (see Figure 2) was adapted from the Rational Number Project interview schedule (Cramer, Behr, Post & Lesh, 1997) It was chosen because it had a perceptual distracter

Task 2, Fraction Pie

Show the student the pie diagram

a) What fraction of the circle is part B? How do you know that?

If unsuccessful go to 5

b) What fraction of the circle is part D? How do you know that?

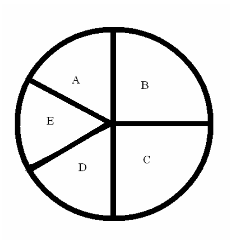


Figure 2 Task 2, Fraction Pie

The third task (originally part of the fractions as a number section) probed for an understanding of fraction density It was not a part-whole question like Task 1 and Task 2 During the testing of the Year 5 children, the task originally asked for a fraction between 3/6 and 4/6, with part (b) the same as above It was decided to change this to between 2/5 and 3/5 as "two and a half fifths" could lead to ½ more easily than finding an equivalent fraction for three and a half sixths

Task 3, Density

Place the card 2/5 to 3/5 in front of the student

a) Can you think of a fraction that is between two-fifths and three-fifths?
If the student says three and a half-fifths ask, What is another name for that?
b) How many fractions are there between two-fifths and three-fifths? Explain how you know

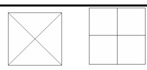
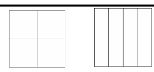
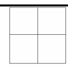
Figure 3 Task 3, Density

Results and Discussion

The results from Task 1, Fold Me A Quarter, were surprising While it was anticipated that thinking of two ways to make quarters and comparing two non-congruent quarters would be difficult for some Year 5 children, it was assumed that it was merely going to be a warm up task for the secondary students In general, when compared to the primary students, the secondary students showed a greater consolidation of fraction concepts, particularly in part-whole understandings, but also more obviously when comparing the size of two written fractions As a group, they performed on the whole better than the mixed ability cohort of Year 5 students In the three tasks discussed in the present paper, the secondary students illustrate, however, that “blind spots” can still occur in the conceptual understanding of fractions beyond the primary years

Two thirds of the children folded the Kinder squares and made quarters in exactly the same way that would later be shown to them in the second part of the question Table 1 shows the number of children who folded each type of quarter and how they subsequently performed on the second part of the task where they were asked to compare the size of two non-congruent quarters offered by the interviewer

Table 1
Types of Quarters Folded and Subsequent Success Comparing Non-Congruent Quarters (Year 5s, n = *18)*

Types of quarters folded by the children using kinder squares			 only
Number of children who folded these quarters in part (a)	12	3	3
Number of children who then answered comparison question (b) correctly	5	0	1
Number of children who then answered comparison question (b) incorrectly	7	3	2

The three Year 10 students answered this task successfully, but two of the Year 8 students (out of five) said that the triangle would give you more While this sample size is too small to draw conclusions about responses of Year 8 students in general, it does alert the researcher to the fact that conservation of area can be unclear to some secondary school children in a fraction context

A similarity between the Year 5 students and the Year 8 students, on the other hand, was in the language that the children used to explain their choice When children chose the triangle as giving more, (none of the 26 students opted for the square quarter as being larger) they described their comparison in terms of spatial visualisation and spatial properties The square could fit into the triangle, for example, or the triangle *looks* bigger/wider, or *is* a larger *shape* Other incorrect explanations used measurement language to describe the comparison between the two quarters, but indicated misconceptions about measurement principles Explanations included that the triangle has diagonal lines not straight lines (which are longer), has three sides that are big, or is on an angle and can fit more in Their language reveals a reliance on a direct comparison; one shape fits inside the other or particular sides are longer Some of the explanations concerning angles and sides may indicate that the children are attending to the attribute of length, rather than area, in order to determine area

One child stated that the triangle covered more amount of area than the square The child was attending to the attribute of length but was unable to compare the two areas of the triangle and the square precisely In some classroom tasks, children may conflate area and length as attending to either property may solve an "area" problem at hand For example, when comparing ribbon (a thick line) or choosing tiles to make a single border, length can substitute for area when working with proportions (Irwin, Vistro-Yu & Ell, 2004) The children who performed poorly on Task 1, Fold Me a Quarter, may benefit from activities that require them to explain when length will also indicate area (a long rectangular ribbon) and when it will not (when comparing two ribbons of different width)

The children who answered correctly that both quarters were the same size, did so without hesitation, the only exception being the child who had only folded his paper in to quarters one way, and he self corrected his first response of the triangle giving more What was more significant, however, was that the language they used to explain their reasoning was completely different to the children who had answered incorrectly The children who answered correctly did not talk about spatial considerations, they saw the pieces as units of a larger whole – as quarters "They're both quarters", was the common response A Year 10 student further qualified that statement by adding that the two kinder squares were the same size *and* that the shaded shapes were both quarters

The responses to this task highlight the need to be certain that some children have the basic principles of measurement, for example conservation of area, that a teacher may assume they have when setting a task Were the children who answered this task incorrectly making pre-memorised shapes when folding the kinder squares and the children who answered correctly been measuring quarters of a unit? Or to put it another way, had the children who correctly compared the two quarters seen them as parts of a larger unit (the whole) that could be re-unitised into non-congruent units (the quarters)?

Success at this task, however, did not predict success at other fraction tasks For one child in Year 5, this task was one of only four he answered correctly, while for another, it was one of 28 tasks he explained correctly So successful use of measurement principles here did not translate in successful use of measurement principles to answer other tasks

Where this task is interesting as a predictor of success at another task is in comparing individual children's responses to Task 1 and Task 2 Task 2, Fraction Pie requires children to identify a fractional part in a part-whole model in the absence of equal parts The two Year 8 students who answered Task 1 incorrectly also answered incorrectly on the second part of Task 2 Instead of calculating that part D was a sixth, they stated that it was a fifth One explained that a fifth was a bit less than a quarter (the part they had correctly identified in part (a) of the question), while the other child counted the number of parts without attending to their relative size Part D was one out of five parts so it was a fifth

This was an occasion where counting and matching would not prove successful The other six secondary students successfully explained why it was a sixth

The Year 5 children generally followed this pattern of being successful at identifying the sixth despite the perceptual distracter if they had successfully compared the two non-congruent quarters (see Table 2) The children who answered correctly could use the language of seeing the half divided into three parts and visualising a mirror image   The children who answered incorrectly had often been able to access a remembered visual image to identify a quarter but then, in part (b), counted the parts and explained that Part D was a fifth Other incorrect answers included; 1/4, 1/2, 4/5 (because D was the fourth letter), 3, and 1/8 (a guess)

Table 2
Relationship Between Performances on Task 1, Fold Me a Quarter, and Task 2, Fraction Pie 2

	Non congruent quarter comparison correct	Non congruent quarter comparison incorrect
Correctly identified 1/6	4	2
Did not identify 1/6	2	10

Children who “saw” shapes could be quite successful with Task 2, Fraction Pie, if only asked to identify a quarter This strategy did not prove successful with 1/6, apparently because it was not a pre-memorised fraction of a circle, and the children who answered this task incorrectly could not efficiently draw on measurement principles to help them answer the question   Children who answered correctly demonstrated in their verbal explanations that they were using measurement principles that they had internalised as fraction understandings They could re-unitise the whole into smaller units that were all equal with no gaps and no overlapping They could use symmetry, reflection and spatial visualisation to support the application of a measurement principle in order to break the whole into measured parts

Task 3, Density is a harder task than the previous two tasks discussed and is asked in a different section of the fraction interview One Year 5 child correctly identified that between two fractions there are as many fractions as you can count One Year 8 child stated that there were an infinite number of fractions, and one Year 10 child explained this by using the term unlimited Interestingly, a different Year 10 student initially said that there would be a few (as in a lot) but when asked to explain his thinking gave equivalent fractions for a half   A half is between 2/5 and 3/5 and there is an infinite number of equivalent fractions that represent a half, but fraction density should not be conflated with equivalence

Only 3 children gave an answer indicating a solid understanding of fraction density Some children were not offered this task, however, if they could not identify a fraction between two fifths and three-fifths in the first part of the question As most children, however, of all age groups found the final part of this task difficult, it is interesting to note the task’s measurement principle components To explain how many fractions there are between two numbers does not involve counting, as there are many possible separate fractions to count That was, however, certainly one strategy employed by the students, Instead, one thinks of the measure between two points, analogous to length, wherein an ever finer unit can measure ever finer graduations

These three tasks illustrate a connection between measurement concepts and a conceptual understanding of fractions As shown by the language the children have used in their answers, such as spatial visualisation (in the pursuit of direct comparisons), or attending to length instead of area, these three tasks highlight the need for further research into the interconnectedness of children's understanding of fractions and children's understandings of the basic principles of measurement These measurement principles include: the effect of choosing different units (the smaller the unit, the more will be needed); iteration of a unit (no gaps and no overlapping); the unit does not have to be congruent (when measuring area, the units do not have to be the same shape); and attending to the attribute to be measured (for example, length or area)

When successfully comparing the two non-congruent quarters, as the children were asked to do in Task 1, Fold Me a Quarter, it may be useful for students to articulate the effect of choosing the unit (they're both quarters, if the triangle were bigger than a square, then you should need less than four triangles to make the whole) They might also articulate the consequences of an iterating unit (they're both quarters, there are no gaps and no overlapping) Also, they might state that the units do not have to be congruent And finally they might articulate the attribute to be measured (the area of the shapes, not the length of the side of a triangle) When identifying one sixth on the fraction pie, children needed to attend to the measurement principle that units must be equal (even if that means ignoring perceptual distracters) The fraction density task highlighted that the continuous nature of rational numbers is different to the discrete counting algorithm possible with whole numbers

Summary and Implications

Notwithstanding the small sample size, overall, the Year 8 and 10 students provided some evidence that a consolidation of a conceptual understanding of fractions happens over time, as they generally performed better on each of the seven sections of the fractions interview than the Year 5 students There were one or two Year 5 students whose individual results were outstanding The three tasks chosen for discussion in this paper, highlight a powerful strategy employed by children who were successful at these tasks The children who were successful at these three tasks, comparing two non-congruent quarters, identifying parts of a whole with perceptual distracters present, and explaining that there is an infinite number of rational numbers between two fractions, made successful use of measurement principles

Some children used measurement and spatial *language* but displayed misconceptions about measurement *principles*, and they were less successful at the tasks The relationship between measurement principles and fractional understanding needs to be explored further with research into the effect of classroom instruction that makes this analogy explicit Also, to justify this, research is needed on mapping children's success at measurement tasks and fraction tasks in a task-based interview that uncovers children's conceptual understanding of both domains The author intends to commence such research

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